1. $F(K,L)$ has $F''_{KK} < 0$, $F''_{LL} < 0$, $F''_{KK} \cdot F''_{LL} - (F''_{LK})^2 > 0$

$$P \cdot F'_L(K,L) = w \qquad (1)$$
$$P \cdot F'_K(K,L) = r \qquad (2)$$

P, w and ~~and~~ r are exogenous and K and L are endogenous. That is to say, both K and L depend on w.

Question: Find $\partial L/\partial w$ and tell something about its sign.

Notice that, given (1) and (2), we do not have $\partial L/\partial w$ as an unknown. Hence, we need to transform both equations in order to have $\partial L/\partial w$ as an unknown and solve for it.

Our first step is to differentiate both (1) and (2) with respect to w:

(Notice that we need to apply the chain rule as we don't directly have either K or L in (1) and (2).)

$$P \cdot (F''_{LK} \cdot \partial K/\partial w + F''_{LL} \cdot \partial L/\partial w) = 1 \qquad (1)'$$
$$P \cdot (F''_{KK} \cdot \partial K/\partial w + F''_{KL} \cdot \partial L/\partial w) = 0 \qquad (2)'$$

Now, dividing both sides by P, we get:

$$F''_{LK} \cdot \partial K/\partial w + F''_{LL} \cdot \partial L/\partial w = 1/P \qquad (1)''$$
$$F''_{KK} \cdot \partial K/\partial w + F''_{KL} \cdot \partial L/\partial w = 0 \qquad (2)''$$

We can now rewrite the system in matrix form to get:

$$\begin{pmatrix} F''_{LL} & F''_{LK} \\ F''_{KL} & F''_{KK} \end{pmatrix} \cdot \begin{pmatrix} \partial L/\partial w \\ \partial K/\partial w \end{pmatrix} = \begin{pmatrix} 1/P \\ 0 \end{pmatrix}$$

which takes the form $A \cdot x = B$. Hence, as $x = A^{-1} \cdot B$, and $A^{-1} = \frac{1}{|A|} \cdot C'$, then $x = \frac{1}{|A|} \cdot C'B$. By finding $|A|$ and C', we get:

$$\begin{pmatrix} \partial L/\partial w \\ \partial K/\partial w \end{pmatrix} = \frac{1}{F''_{LL} \cdot F''_{KK} - (F''_{LK})^2} \cdot \begin{pmatrix} F''_{KK} & -F''_{LK} \\ -F''_{KL} & F''_{LL} \end{pmatrix} \begin{pmatrix} 1/P \\ 0 \end{pmatrix} \qquad (3)$$

By multiplying the matrices, we get:

$$\begin{pmatrix} \partial L/\partial w \\ \partial K/\partial w \end{pmatrix} = \frac{1}{F''_{LL} \cdot F''_{KK} - (F''_{LK})^2} \cdot \begin{pmatrix} F''_{KK}/P \\ -F''_{KL}/P \end{pmatrix} \Rightarrow \boxed{\partial L/\partial w = \frac{F''_{KK}}{P \cdot (F''_{LL} \cdot F''_{KK} - (F''_{LK})^2)}}$$

As $F''_{KK} < 0$ and $F''_{LL} \cdot F''_{KK} - (F''_{LK})^2 > 0$, then $\boxed{\partial L/\partial w < 0}$

2) $f(x,y) = 8x^3 - 12xy + y^3$

a) Find all the stationary points and ~~find~~ classify them:

To find them, we find the first derivatives and equal them to 0:

- $\frac{\partial f(x,y)}{\partial x} = 0 \Rightarrow 24x^2 - 12y = 0 \Rightarrow 24x^2 = 12y \Rightarrow \boxed{2x^2 = y}$ (1)

- $\frac{\partial f(x,y)}{\partial y} = 0 \Rightarrow 3y^2 - 12x = 0 \Rightarrow 3y^2 = 12x \Rightarrow \boxed{y^2 = 4x}$ (2)

By squaring both sides of (1), we get:

$$\boxed{4x^4 = y^2} \quad (1)'$$

By equalling (1)' and (2), we get:

$$4x^4 = 4x \Rightarrow \boxed{x^4 = x}$$

From which there are only two answers: $x = 0$ or $x = 1$. Hence,

If $x = 0 \Rightarrow$ by (1), $2(0)^2 = y \Rightarrow y = 0$
If $x = 1 \Rightarrow$ by (1), $2(1)^2 = y \Rightarrow y = 2$

We have two stationary points: $(0,0)$ and $(1,2)$

To classifffy them, we need to take the second order derivatives:

$\partial^2 f(x,y)/\partial x^2 = 48x$
$\partial^2 f(x,y)/\partial y^2 = 6y$
$\partial^2 f(x,y)/\partial x \partial y = -12$

$\rightarrow F''_{xx} \cdot F''_{yy} - (F''_{xy})^2 = 48x \cdot 6y - 144$

Notice that both x and y can take any value from $-\infty$ to $+\infty$. Hence, $\partial^2 f(x,y)\partial x^2$ and $\partial^2 f(x,y)/\partial y^2$ need to be ~~again~~ examined close to the stationary points in order to determine local concavity or convexity at that point:

If $(0,0) \Rightarrow \partial^2 f(x,y)/\partial x^2 = 0 = \partial^2 f(x,y)/\partial y^2$; $F''_{xx} \cdot F''_{yy} - (F''_{xy})^2 = -144 < 0$.
Hence, $(0,0)$ is a saddle point.

If $(1,2) \Rightarrow \partial^2 f(x,y)/\partial x^2 = 48(1) = 48 > 0$
$\partial^2 f(x,y)/\partial y^2 = 6(2) = 12 > 0$
$F''_{xx} \cdot F''_{yy} - (F''_{xy})^2 = 48 \cdot 12 - 144 = 432 > 0$.
Hence, $(1,2)$ is a local minimum.

b) In order to analyse whether our stationary points can be a global maximum or minimum we need to check the signs of F''_{xx} and F''_{yy} for the whole domain of the function, which ranges from $-\infty$ to $+\infty$.
A function will be a global maximum if $F''_{xx} \leq 0$ and $F''_{yy} \leq 0$ for all the values of (x, y) and a global minimum if $F''_{xx} \geq 0$ and $F''_{yy} \geq 0$ for all the values of (x, y).

- As $\partial^2 f(x,y)/\partial x^2 = 48x$, it will be positive if $x > 0$ and negative if $x < 0$
- As $\partial^2 f(x,y)/\partial y^2 = 6y$, It will be positive if $y > 0$ and negative if $y < 0$.

Hence, as F''_{xx} and F''_{yy} won't be either positive or negative for all the possible values of x and y, the function does not have global maximum or minimum.

c) Consider $g(x, y) = (8x^3 - 12xy + y^3)^2$. Does it have a global max. or min.?

→ First, notice that $a^2 \geq 0$. Hence, $g(x, y) \geq 0$ for all the values of x and y. It follows, then, that all x, y such that
$8x^3 - 12xy + y^3 = 0$ will be global minimum. One such point was found in the previous sections: $(0,0)$.

→ Second, notice that, if $x = 0$, $g(x, y) = y^6$. Let's consider any point (x_0, y_0) such that $g(x_0, y_0)$. We can always find a y such that $g(0, y) > g(x_0, y_0)$, as $\lim_{y \to \pm\infty} g(x, y) = \infty$. Hence, there is no global maximum.

(3) $f(x,y) = e^{x+y} + e^{x-y} - \frac{3}{2}x - \frac{1}{2}y$.

a) Show that $f(x,y)$ is convex.

Remember that, for $f(x,y)$ to be convex, $f''_{xx}(x,y) \geq 0$, $f''_{yy}(x,y) \geq 0$ and $f''_{xx} \cdot f''_{yy} - (f''_{xy})^2 \geq 0$.

$$f'_x(x,y) = e^{x+y} + e^{x-y} - \frac{3}{2}$$

$$f'_y(x,y) = e^{x+y} - e^{x-y} - \frac{1}{2}$$

$$f''_{xx}(x,y) = e^{x+y} + e^{x-y}$$

$$f''_{yy}(x,y) = e^{x+y} + e^{x-y}$$

$$f''_{xy}(x,y) = -e^{x-y} + e^{x+y}$$

- $\Rightarrow f''_{xy}(x,y) = (e^{x+y} - e^{x-y})^2 = e^{2(x+y)} + e^{2(x-y)} - 2e^{x+y} \cdot e^{x-y} =$
$= \boxed{f''_{xy}(x,y) = e^{2(x+y)} + e^{2(x-y)} - 2 \cdot e^{2x}}$ (1)

- $f''_{xx} \cdot f''_{yy} = (e^{x+y} + e^{x-y})^2 = e^{2(x+y)} + e^{2(x-y)} + 2 \cdot e^{x+y} \cdot e^{x-y} =$
$= \boxed{f''_{xx} \cdot f''_{yy} = e^{2(x+y)} + e^{2(x-y)} + 2 \cdot e^{2x}}$ (2)

Hence, $f''_{xx} \cdot f''_{yy} - (f''_{xy})^2 = (2) - (1)$

$\Rightarrow f''_{xx} \cdot f''_{yy} - (f''_{xy})^2 = e^{2(x+y)} + e^{2(x-y)} + 2 \cdot e^{2x} - e^{2(x+y)} - e^{2(x+y)} + 2 \cdot e^{2x}$
$= \boxed{4 \cdot e^{4x}}$

Now, we can evaluate the sign of the three relevant equations:

- $f''_{yy} = f''_{xx} = e^{x+y} + e^{x-y} \geq 0$.

Note that $e^{\infty} = \infty$ and $e^{-\infty} = \frac{1}{e^{\infty}} = 0$. Hence, no matter the values of x and y, f''_{xx} and f''_{yy} will always be ≥ 0.

- $f''_{xx} \cdot f''_{yy} - (f''_{xy})^2 = 4 \cdot e^{4x}$. By a similar argument, it doesn't matter the value of x, $4 \cdot e^{4x}$ will always be ≥ 0.

$\boxed{\text{Hence, } f(x,y) \text{ is convex}}$

b) Find the minimum of $f(x,y)$

The first step is to equal the first order derivatives to 0 in order to find the stationary points:

$$f'_x = 0 \Rightarrow e^{x+y} + e^{x-y} = \frac{3}{2} \qquad (1)$$

$$f'_y = 0 \Rightarrow e^{x+y} - e^{x-y} = \frac{1}{2} \qquad (2)$$

- Isolate e^{x-y} in both (1) and (2):

$$e^{x-y} = \frac{3}{2} - e^{x+y} \qquad (1)'$$

$$e^{x-y} = e^{x+y} - \frac{1}{2} \qquad (2)'$$

- Set $(1)' = (2)'$:

$$\frac{3}{2} - e^{x+y} = e^{x+y} - \frac{1}{2} \Rightarrow 2 = 2e^{x+y} \Rightarrow \boxed{1 = e^{x+y}} \quad (3).$$

- Take logs in both hand sides of (3):

$$\ln(1) = \ln(e^{x+y}) \Rightarrow 0 = x + y \Rightarrow \boxed{x = -y} \quad (4)$$

- Substitute (4) in (1):

$$e^{-y+y} + e^{-y-y} = \frac{3}{2} \Rightarrow e^{0} + e^{-2y} = \frac{3}{2} \Rightarrow$$

$$\Rightarrow e^{-2y} = \frac{3}{2} - 1 \Rightarrow \boxed{e^{-2y} = \frac{1}{2}} \quad (5).$$

- Take logs in both hand sides of (5):

$$\ln(e^{-2y}) = \ln(1/2) \Rightarrow -2y = \ln\left(\tfrac{1}{2}\right) \Rightarrow \boxed{y = \frac{\ln(1/2)}{2}} \quad (6)$$

- Substitute (6) in (4) to get: ~~y~~ $x = -\frac{\ln(1/2)}{2}$.

Hence, $\left(-\frac{\ln(1/2)}{2}, \frac{\ln(1/2)}{2}\right)$ is a stationary point, and, as $f(x,y)$ is convex, then it follows that the stationary point is a global minimum.

4 $F(K, L) = (L^{1/2} + K^{1/2})^2$

$2 \cdot L + 3 \cdot K$ are the costs of production

a) In what proportion should the firm combine capital and labour to minimize cost.

One way to proceed is to minimize $2 \cdot L + 3 \cdot K$ s.t. $Q = (L^{1/2} + K^{1/2})^2$, find the optimal L and K and then compute L/K or K/L. In order to do that, let's first use the substitution method and substitute the constraint in the objective function $(2 \cdot L + 3 \cdot K)$. But, before that, let's first isolate L:

- $Q = (L^{1/2} + K^{1/2})^2 \Rightarrow Q^{1/2} = \left((L^{1/2} + K^{1/2})^2\right)^{1/2} \Rightarrow Q^{1/2} = L^{1/2} + K^{1/2} \Rightarrow$

$\Rightarrow Q^{1/2} - K^{1/2} = L^{1/2} \Rightarrow \boxed{L = (Q^{1/2} - K^{1/2})^2}$ (1).

- Substitute (1) in $2L + 3K$ in order to get:

$f(K) = 2 \cdot (Q^{1/2} - K^{1/2})^2 + 3K$.

- Find $f'(K) = 0$ in order to get the optimal K:

$f'(K) = 0 \Rightarrow \boxed{(4)} \cdot (Q^{1/2} - K^{1/2}) \cdot \boxed{\left(-\frac{1}{2}\right)} (K^{-1/2}) + 3 = 0 \Rightarrow$

$\Rightarrow -2 (Q^{1/2} - K^{1/2})(K^{-1/2}) + 3 = 0 \Rightarrow (-2)\left(Q^{1/2} \cdot K^{-1/2} - K^{(1/2 - 1/2)}\right) + 3 = 0 \Rightarrow$

$\Rightarrow -2 Q^{1/2} \cdot K^{-1/2} + 2 \cdot K^0 + 3 = 0 \Rightarrow 5 = 2 \cdot \frac{Q^{1/2}}{K^{1/2}} \Rightarrow$

$\Rightarrow 5 K^{1/2} = 2 Q^{1/2} \Rightarrow \boxed{K^{1/2} = \frac{2}{5} \cdot Q^{1/2}} \Rightarrow \boxed{K = \frac{4}{25} Q}$

- Substitute $K^{1/2}$ in (1):

$L = (Q^{1/2} - \frac{2}{5} Q^{1/2})^2 \Rightarrow L^{1/2} = Q^{1/2} - \frac{2}{5} Q^{1/2} \Rightarrow$

$\Rightarrow L^{1/2} = \frac{3}{5} Q^{1/2} \Rightarrow \boxed{L = \frac{9}{25} Q}$

- Find K/L:

$\frac{\frac{4}{25} \cdot Q}{\frac{9}{25} \cdot Q} = \boxed{\frac{4}{9}}$.

Another way to proceed is to use the lagrangian method:

$$\mathcal{L} = 3K + 2L - \lambda\left[(L^{1/2} + K^{1/2})^2 - Q \right]$$

- $\dfrac{\partial \mathcal{L}}{\partial L} = 0 \Rightarrow 2 - \lambda \cdot 2 \cdot (L^{1/2} + K^{1/2}) \left(\frac{1}{2}\right) \cdot L^{-1/2} = 0 \Rightarrow$

$\Rightarrow 2 - \lambda \cdot (L^{1/2} + K^{1/2}) \cdot L^{-1/2} = 0 \Rightarrow$

$\Rightarrow 2 = \lambda (L^{1/2} + K^{1/2}) \cdot L^{-1/2} \Rightarrow$

$\Rightarrow \boxed{\dfrac{2}{(L^{1/2} + K^{1/2}) \cdot L^{-1/2}} = \lambda}$ (A)

- $\dfrac{\partial \mathcal{L}}{\partial K} = 0 \Rightarrow 3 - \lambda \cdot 2 \cdot (L^{1/2} + K^{1/2}) \cdot \frac{1}{2} \cdot K^{-1/2} = 0 \Rightarrow$

$\Rightarrow 3 = \lambda (L^{1/2} + K^{1/2}) \cdot K^{-1/2} \Rightarrow$

$\Rightarrow \boxed{\dfrac{3}{(L^{1/2} + K^{1/2}) K^{-1/2}} = \lambda}$ (B)

- setting (A) = (B), we get:

$$\frac{2}{(L^{1/2} + K^{1/2}) \cdot L^{-1/2}} = \frac{3}{(L^{1/2} + K^{1/2}) \cdot K^{-1/2}} \Rightarrow \frac{2}{3} = \frac{\boxed{(L^{1/2} + K^{1/2})} \cdot L^{-1/2}}{\boxed{(L^{1/2} + K^{1/2})} \cdot K^{-1/2}} \Rightarrow$$

$$\Rightarrow \frac{2}{3} = \frac{L^{-1/2}}{K^{-1/2}} \Rightarrow \frac{2}{3} = \frac{K^{1/2}}{L^{1/2}} \Rightarrow \frac{2}{3} = \left(\frac{K}{L}\right)^{1/2} \Rightarrow \frac{K}{L} = \left(\frac{2}{3}\right)^2 = \boxed{\frac{4}{9} = \frac{K}{L}}$$

b) Find the cost function, defined as the minimum cost of producing Q units.

From part a), we know that $K^* = \frac{4}{25} Q$ and $L^* = \frac{9}{25} Q$.

We know that the cost of production is defined as:

$$C(Q) = 2L + 3K.$$

Hence, to find the minimum cost, we need to ~~find~~ substitute in $C(Q)$ the L and K that minimise that function:

$$C(Q^*) = 2 \cdot L^* + 3 \cdot K^* \Rightarrow$$

$$\Rightarrow C(Q^*) = 2 \cdot \left(\frac{9}{25} Q^*\right) + 3\left(\frac{4}{25} Q^*\right) = \frac{18}{25} Q^* + \frac{12}{25} Q^* = \frac{30}{25} Q^* = \boxed{\frac{6}{5} Q^*}$$

Hence, $\boxed{C(Q^*) = 1.2 \cdot Q^*}$.

5 $F(K,L,T) = K^{1/3} \cdot L^{1/3} \cdot T^{1/3}$

$C(Q) = K + 2L + 4T$

As the firm wants to produce $Q = 10$, then:

$$F(K,L,T) = 10 = K^{1/3} \cdot L^{1/3} \cdot T^{1/3} \Rightarrow \frac{10}{K^{1/3} \cdot L^{1/3}} = T^{1/3} \Rightarrow \boxed{T = \frac{1000}{KL}} \quad (A)$$

As before, we can solve this by either substitution or the lagrangian method. Let's, this time, only use the substitution method.

- Substitute (A) into $C(Q)$ to get:

$$C(Q) = K + 2L + \frac{4000}{KL}$$

- As with every function of several variables, to find a stationary point we need to find the first derivatives and equal them to 0:

$$C'_K = 1 - \frac{4000L}{K^2L^2} = 0 \Rightarrow 1 = \frac{4000L}{K^2L^2} \Rightarrow \boxed{K^2L^2 = 4000L} \quad (1)$$

$$C'_L = 2 - \frac{4000K}{K^2L^2} = 0 \Rightarrow 2 = \frac{4000K}{K^2L^2} \Rightarrow \boxed{K^2L^2 = 2000K} \quad (2)$$

- Equalling (1) and (2), we get:

$$4000L = 2000K \Rightarrow K = \frac{4000}{2000}L \Rightarrow \boxed{K = 2L} \quad (3)$$

- Rearranging (1) by dividing both hand sides by L, we get:

$$\frac{K^2L^2}{L} = 4000 \Rightarrow \boxed{K^2L = 4000}.$$

- Substituting $K = 2L$, we get:

$$(2L)^2 \cdot L = 4000 \Rightarrow 4L^2 \cdot L = 4000 \Rightarrow$$
$$\Rightarrow L^3 = 1000 \Rightarrow L^3 = 10^3 \Rightarrow (L^3)^{1/3} = (10^3)^{1/3} \Rightarrow \boxed{L = 10} \quad (4)$$

- Substituting (4) in (3), we get: $K = 2(10) \Rightarrow \boxed{K = 20} \quad (5)$

- Substituting (4) and (5) in (A), we get:

$$T = \frac{1000}{(20)(10)} = \frac{1000}{200} \Rightarrow \boxed{T = 5} \quad (6)$$

Hence, the stationary point is $(20, 10, 5)$

In order to check that it is a minimum, we find the second order derivatives:

$$C''_{KK} = \frac{4000L^3 \, 2K}{K^4L^4} = \boxed{\frac{8000}{K^3L} > 0} \quad (\text{As } K \text{ and } L \geq 0)$$

$$C''_{LL} = \frac{4000K^3 \, 2L}{K^4L^4} = \boxed{\frac{8000}{KL^3} > 0} \quad (\text{As } K \text{ and } L \geq 0)$$

$$C''_{LK} = \frac{-4000K^2L^2 + 4000K \, 2LK^2}{K^4L^4} = \frac{-4000}{K^2L^2} + \frac{8000K^2L^2}{K^4L^4} \Rightarrow$$

$$= \frac{-4000+8000}{K^2L^2} = \frac{4000}{K^2L^2}$$

$$\Rightarrow \boxed{(C''_{LK})^2 = \frac{16000000}{K^4L^4}}$$

$$\Rightarrow C''_{KK} \cdot C''_{LL} - (C''_{LK})^2 = \frac{64000000}{K^4L^4} - \frac{16000000}{K^4L^4} = \boxed{\frac{48000000}{K^4L^4} > 0}$$

As $K, L \geq 0$. Hence, the function is convex and $(20, 10, 5)$ is a global minimum.

6 $f(x,y,z) = x^2 + x + y^2 + z^2$

a) $x^2 + 2y^2 + 2z^2 = 16$ is the constraint. Find the maximum and minimum of $f(x,y,z)$

- Notice that, if we isolate y^2+z^2 in the constraint, we can solve the problem by substitution turning our objective function into a function of a single variable!

$$16 - x^2 = 2\cdot(y^2+z^2) \Rightarrow \boxed{8 - \frac{1}{2}x^2 = y^2 + z^2} \quad (A)$$

Hence,

$$f(x,y,z) = x^2 + x + 8 - \frac{1}{2}x^2 \Rightarrow \boxed{f(x) = \frac{1}{2}x^2 + x + 8} \quad (1)$$

Finding $f'(x) = 0$, we get:

$$f'(x) = 0 \Rightarrow x + 1 = 0 \Rightarrow \boxed{x = -1} \quad (2)$$

- Substituting (2) into (A), we get:

$$8 - (1/2)(-1)^2 = y^2 + z^2 \Rightarrow 8 - 1/2 = y^2 + z^2 \Rightarrow \boxed{7.5 = y^2 + z^2}$$

Hence, $x = -1$ and any (y, z) such that $y^2 + z^2 = 7.5$ are our minima. because $f''(x) = 1 > 0$ and, hence, $f(x)$ is convex.

For finding the maximum, note that, in (A), $y^2 + z^2 \geq 0$ as any number squared has to be positive. Hence, $8 - \frac{1}{2}x^2 \geq 0 \Rightarrow 16 \geq x^2 \Rightarrow \boxed{\pm 4 \geq x}$. Hence, $-4 \leq x \leq 4$ for (A) to hold true. Finding $f(x=4)$, $f(x=-4)$, we get:

$$f(x=-4) = (\tfrac{1}{2})(16) + 8 - 4 = 12$$
$$f(x=+4) = (1/2)(16) + 8 + 4 = 20.$$

Therefore, $(4,0,0)$ is the maximum.

b) $x^2 + 2y^2 + 2z^2 \leq 16$ is the constraint. Find the maximum and minimum of $f(x, y, z)$

As we don't have a strict inequality, we need to find the stationary points at the interior and the points at the boundary of the constraint.

In the interior:

$f'_x(x, y, z) = 0 \Rightarrow 2x + 1 = 0 \Rightarrow x = -1/2$

$f'_y(x, y, z) = 0 \Rightarrow 2y = 0 \Rightarrow y = 0$

$f'_z(x, y, z) = 0 \Rightarrow 2z = 0 \Rightarrow z = 0$

Hence, the unique stationary point at the interior is $(-1/2, 0, 0)$.
As found before, in the boundary the minimum is $(x = -1, y^2 + z^2 = 7.5)$ and the maximum is $(4, 0, 0)$.
Now, finding the value of the function at the three points, we get:

$f(-1/2, 0, 0) = (-\frac{1}{2})^2 + (-1/2) + 0^2 + 0^2 = \frac{1}{4} + (-\frac{1}{2}) = \boxed{-\frac{1}{4}}$.

$f(-1, y^2 + z^2 = 7.5) = (-1)^2 + (-1) + 7.5 = 1 - 1 + 7.5 = \boxed{7.5}$

$f(4, 0, 0) = 4^2 + 4 + 0^2 + 0^2 = \boxed{20}$

Hence, $(-1/2, 0, 0)$ is the minimum and $(4, 0, 0)$ is the maximum.

7 max $x+2y$ s.t. $x^{1/2}+y^{1/2}=30$

a) • write the lagrangian:

$$\mathcal{L}=x+2y-\lambda(x^{1/2}+y^{1/2}-30)$$

A) $\frac{\partial \mathcal{L}}{\partial x}=0 \Rightarrow 1-\frac{1}{2}\lambda x^{-1/2}=0 \Rightarrow 1=\lambda\frac{1}{2}\cdot x^{-1/2} \Rightarrow 1=\lambda\cdot\frac{1}{2x^{1/2}} \Rightarrow$

$\Rightarrow \boxed{2x^{1/2}=\lambda}$ (1)

B) $\frac{\partial \mathcal{L}}{\partial y}=0 \Rightarrow 2-\lambda(\frac{1}{2})y^{-1/2}=0 \Rightarrow 2=\lambda\cdot\frac{1}{2}\cdot y^{-1/2} \Rightarrow 2=\lambda\cdot\frac{1}{2y^{1/2}} \Rightarrow$

$\Rightarrow \boxed{4y^{1/2}=\lambda}$ (2)

• Equal (1) = (2):

$2x^{1/2}=4y^{1/2} \Rightarrow \boxed{x^{1/2}=2y^{1/2}}$ (3)

C) $\frac{\partial \mathcal{L}}{\partial \lambda}=0 \Rightarrow -x^{1/2}-y^{1/2}+30=0 \Rightarrow \boxed{x^{1/2}+y^{1/2}=30}$ (4)

• Substitute (3) in (4) to get:

$2y^{1/2}+y^{1/2}=30 \Rightarrow 3y^{1/2}=30 \Rightarrow y^{1/2}=10 \Rightarrow \boxed{y=100}$ (5)

• Substitute (5) in (3) to get:

$x^{1/2}=2\cdot(100)^{1/2} \Rightarrow (x^{1/2})^2=(2\cdot(100)^{1/2})^2 \Rightarrow x=4\cdot 100 \Rightarrow$

$\Rightarrow \boxed{x=400}$. (6).

Hence, from the lagrangian, we get: $\boxed{x^*=400,\ y^*=100}$

But, note that $f(x=400, y=100)=(400)+2(100)=400+200=600$

whilst $f(x=0, y=900) \Rightarrow (0)+(2)(900)=1800$

Hence, $\boxed{f(0,900)>f(400,100)}$ and $0^{1/2}+(900)^{1/2}=30$. Hence, not only $(0,900)$ gives a higher value but it also fulfills the constraint!.

b) The constraint is $x^{1/2} + y^{1/2} = 30 \Rightarrow$

$$\Rightarrow \boxed{y^{1/2} = 30 - x^{1/2}} \quad (1)$$

- Let's find values of y and x that make (1) true:

Ⓒ

y	x
900	0
0	900
100	400
625	25
25	625

→ As we can see,:

i) the constraint is non linear

ii) Passes through (100, 400)

iii) (900,0) and (0,900) are the extremes of the constraint.

- Let's find some of the level curves of $x + 2y$.

$x + 2y = 100$

$\boxed{y = 50 - \frac{1}{2}x}$ (A)

$x + 2y = 200$

$\boxed{y = 100 - \frac{1}{2}x}$ (B)

$x + 2y = 400$

$\boxed{y = 300 - \frac{1}{2}x}$ (C)

$x + 2y = 1800$

$\boxed{y = 900 - \frac{1}{2}x}$ (D)

As we can see, all are straight lines, so let's just find the points of the x- and y- axis through which these lines pass:

(A)

y	x
0	100
50	0

(B)

y	x
0	200
100	0

$100 = x - \frac{1}{2}(400) \Rightarrow$

$\Rightarrow 100 + \frac{1}{2}(400) = x \Rightarrow$

$\Rightarrow 100 + 200 = \boxed{300} \longrightarrow \boxed{y = 300 - \frac{1}{2}x}$

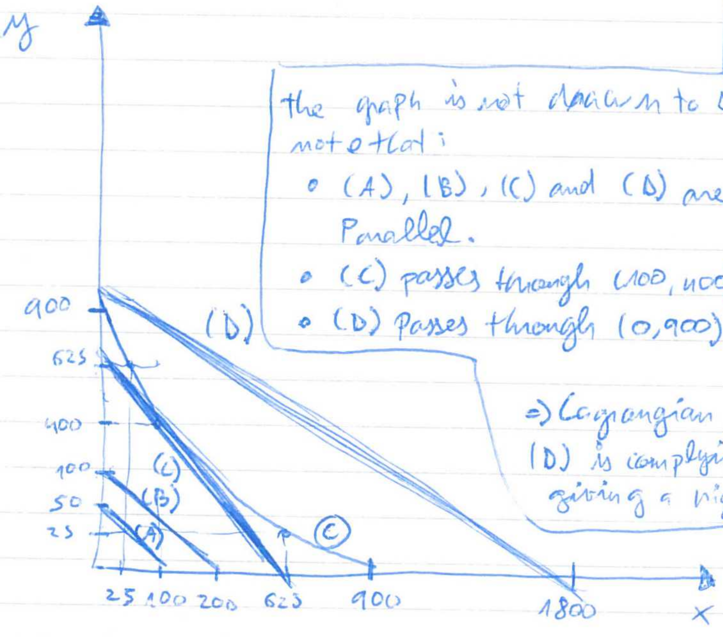
this line passes through (100, 400)

(C)

y	x
300	0
0	600
100	400

(D)

y	x
900	0
0	1800



The graph is not drawn to scale, but if it were, note that:

- (A), (B), (C) and (D) are level curves and, hence, parallel.
- (C) passes through (100, 400) and is a tangent to Ⓒ.
- (D) passes through (0, 900) although is not tangent to Ⓒ.

⇒ Lagrangian finds the tangency, but (D) is complying with the restriction while giving a higher output