$$D(Q) = b - a(q_1 + q_2)$$

$$TC(q) = c_i q$$

1. Cournot:

1.1. Optimal Quantities:

$$\pi_1(q_1, q_2) = \big(b - a(q_1 + q_2)\big) \times q_1 - c_1 q_1$$

$$\pi_1(q_1, q_2) = bq_1 - aq_1^2 - aq_1q_2 - c_1q_1$$

<u>Firm 1 maximises income:</u>

$$\frac{\partial \pi_1(q_1, q_2)}{\partial q_1} = 0 \rightarrow b - 2aq_1 - aq_2 - c_1 = 0$$

Isolating q_1 in the RHS, we get:

$$b - aq_2 - c_1 = 2aq_1$$

Taking q_1 as a common factor in the RHS, we can rewrite the previous expression as:

$$\frac{b - aq_2 - c_1}{2a} = q_1^*$$

Firm 2's income is defined by:

$$\pi_2(q_1, q_2) = \big(b - a(q_1 + q_2)\big) \times q_2 - c_2 q_2$$

$$\pi_2(q_1, q_2) = bq_2 - aq_1q_2 - aq_2^2 - c_2q_2$$

Firm 2 maximises income:

$$\frac{\partial \pi_2(q_1, q_2)}{\partial q_1} = 0 \rightarrow b - aq_1 - 2aq_2 - c_2 = 0$$

Isolating q_1 in the RHS, we get:

$$b - aq_1 - c_2 = 2aq_2$$

Taking q_1 as a common factor in the RHS, we can rewrite the previous expression as:

$$\frac{b - aq_1 - c_2}{2a} = q_2^*$$

Substitute q_2 into q_1 to get:

$$q_1 = \frac{b - a\left(\frac{b - aq_1 - c_2}{2a}\right) - c_1}{2a}$$

$$2aq_1 = b - a\left(\frac{b - aq_1 - c_2}{2a}\right) - c_1$$

$$2aq_1 = b - \left(\frac{b - aq_1 - c_2}{2}\right) - c_1$$

$$2aq_1 = \frac{2b - b + aq_1 + c_2 - 2c_1}{2}$$

$$4aq_1 = 2b - b + aq_1 + c_2 - 2c_1$$

$$4aq_1 - aq_1 = 2b - b + c_2 - 2c_1$$

$$3aq_1 = b + c_2 - 2c_1$$

$$q_1^* = \frac{b + c_2 - 2c_1}{3a}$$

Substituting the value of q_1 into the expression of q_2 found earlier, we get:

$$\frac{b - a \times \left(\frac{b + c_2 - 2c_1}{3a}\right) - c_2}{2a} = q_2$$

Simplifying, we get:

$$\frac{b - \left(\frac{b + c_2 - 2c_1}{3}\right) - c_2}{2a} = q_2$$

By multiplying b and c_2 by $\frac{3}{3} = 1$, we can rewrite the previous expression as:

$$\frac{\left(\frac{3b - b - c_2 + 2c_1 - 3c_2}{3}\right)}{2a} = q_2$$

Which, if simplifying, we get:

$$\frac{\left(\frac{2b - 4c_2 + 2c_1}{3}\right)}{2a} = q_2$$

Taking 2 as a common factor in the numerator, we get:

$$\frac{\left(\frac{2 \times (b - 2c_2 + c_1)}{3}\right)}{2a} = q_2$$

Noting that $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$, we can rewrite the previous expression as:

$$\frac{2 \times (b - 2c_2 + c_1)}{3 \times 2a} = q_2$$

Simplifying, we get:

$$\frac{b - 2c_2 + c_1}{3a} = q_2^*$$

1.1.1. Optimal quantities: comparative statics

The optimal quantities of firm 1 and firm 2 are:

$$q_1^* = \frac{b + c_2 - 2c_1}{3a}$$

$$\frac{b - 2c_2 + c_1}{3a} = q_2^*$$

1.1.1.1. Demand shock:

An increase in b (positive demand shock) will increase the quantities produced of both firms. This is intuitive: the more demand there is the more the firms will produce.

$$\frac{\partial q_1^*}{\partial b} = \frac{\partial q_2^*}{\partial b} = \frac{1}{3a} \geq 0$$

1.1.1.2. Price sensitivity shock:

An increase in the slope of the demand function (that is, how responsive consumers are to an increase in price – if a is big, consumers' reduction in consumption for a one unit increase in price will be big –) will reduce the quantities produced by both firms. This result states that the more sensitive consumers are to price changes the lower the quantity supplied in equilibrium:

$$\frac{\partial q_1^*}{\partial a} = -\frac{b + c_2 - 2c_1}{3a^2} \leq 0$$

$$\frac{\partial q_2^*}{\partial a} = -\frac{b - 2c_2 + c_1}{3a^2} \leq 0$$

1.1.1.3. Competitor's cost shock:

An increase in the cost of the competitor firm will increase the quantity produced by your firm. This result is intuitive: the more costly it is for the other firm to produce the same product, the higher the price they will have to charge in order not to make losses. This makes you a relatively more attractive option, as the goods sold are homogeneous. Hence, it is logical to assume that an increase in the cost of the other firm will increase the quantity you can supply in equilibrium:

$$\frac{\partial q_1^*}{\partial c_2} = \frac{\partial q_2^*}{\partial c_1} = \frac{1}{3a} \geq 0$$

1.1.1.4. Own cost shock:

An increase in your own cost of production will reduce the quantity you supply in equilibrium. This result is also intuitive: the more costly it is for you to produce a given product the more you'll have to charge in order not to make losses. Hence, it is logical to expect that, as a price setter, you'll have incentives to increase the final price. This will have the obvious consequence of reducing the quantity you supply in equilibrium:

$$\frac{\partial q_1^*}{\partial c_1} = \frac{\partial q_2^*}{\partial c_2} = -\frac{2}{3a} \leq 0$$

1.1.1.5. Joint demand and competitor's cost shocks:

Another result we get from this comparative statics is that the effect of an equivalent change in a positive demand shock (b) and an increase in your competitor's unit cost of production should alter your quantity supplied equivalently. Another way to put it is the following one: if the positive demand shock arrives together with a decrease in your competitor's unit cost of production, and the magnitude of both effects is the same, you shouldn't expect your quantity supplied in equilibrium to vary, as one effect cancels out the other.

1.1.1.6. Joint own and competitor's cost shocks:

Another result is comparing the effects of an increase in your unit cost of production vs. an increase in your competitor's unit cost of production. The magnitude of the effect in your unit cost of production is double the effect of an increase in your competitor's cost of production. In practical terms, this means that, in order for your quantity supplied not to change in equilibrium, per one unit increase in your unit cost of production your competitor's unit cost of production should raise by 2 units.

1.1.1.7. Joint price sensitivity and either own or competitor's cost shocks:

Finally, note that there is an interaction between the sensitive of consumer's to price changes and the change in unit costs of production. Whenever the sensitivity of consumer's to prices increases, the effect of an increase in either your or our competitor's unit cost of production decreases:

$$\frac{\partial^2 q_1^*}{\partial c_1 \partial a} = \frac{\partial^2 q_2^*}{\partial c_2 \partial a} = +\frac{2}{3a^2} \geq 0$$

$$\frac{\partial^2 q_1^*}{\partial c_2 \partial a} = \frac{\partial^2 q_2^*}{\partial c_1 \partial a} = -\frac{1}{3a^2} \leq 0$$

1.2. Optimal Price:

To find the price, substitute the expressions we found for q_1 and q_2 into $D(Q)$ to get:

$$D(Q) = b - a\left(\frac{b + c_2 - 2c_1}{3a} + \frac{b - 2c_2 + c_1}{3a}\right)$$

Which can be rewritten as:

$$D(Q) = b - a\left(\frac{b + c_2 - 2c_1 + b - 2c_2 + c_1}{3a}\right)$$

Which simplifies to:

$$D(Q) = b - a\left(\frac{2b - c_2 - c_1}{3a}\right)$$

Which is equivalent to:

$$D(Q) = b - \frac{2b - c_2 - c_1}{3}$$

Multiplying the first element by $\frac{3}{3} = 1$, we get:

$$D(Q) = \frac{3b - 2b + c_2 + c_1}{3}$$

Which, after simplifying, becomes:

$$D(Q) = \frac{b + c_2 + c_1}{3} = p^*$$

1.2.1. Optimal price: comparative statics

$$D(Q) = \frac{b + c_2 + c_1}{3} = p^*$$

1.2.1.1. Positive demand, own cost and competitor's cost shocks:

Note that a change in either variable influences price positively.

First, a positive demand shock increases the equilibrium price. This shouldn't surprise us: as in normal competitive equilibrium, an increase in demand (b) will increase both the quantity supplied (see the previous comparative statis) and the optimal price. Second, an increase in either your unit cost of production or your competitor's cost of production will increase the equilibrium price. This is intuitive.

Remember that, unlike in perfect competitive equilibrium, in oligopolistic or monopolistic markets the firms act as price setters. So, whenever their unit cost of production goes up they have the power to decide to influence the price upwards to cover for potential losses.

Notes: This is just the theory, and need not be what happens in real life. People may not choose to act rationality if we understand maximisation of own payoff as the definition of rationality. A firm that cares about their fellow citizens (e.g. an altruistic entrepreneur) may not necessarily want to increase price (or, at least, not as much as he would have done would have he or she behaved rationally) when they face an increase in cost. This may be due for a variety of reasons. Imagine a situation of a pandemic (quite closely related to today's problem). This inevitably impacts people by reducing their income or even losing their jobs. The enterprises normally have more resouces, as they have easier access to credit. So, a firm caring by their fellow citizens may not want to let people bear the cost of a situation (e.g. a pandemic) that worsens both the consumers and the firms. Doing so may be seen as immoral, as someone may see the role of the firm of having a duty to be fair and to act in accordance to what's the most reasonable, and not rational, action. So, even when this is what the theory assumes, don't learn this by hard or assume this is reality: reality is normally much more complex that what maths suggest. After all, you are capturing society, and social factors may be as important as purely economic ones.

1.3. Optimal profit:

The profit of both firms is driven by:

$$\pi_1(q_1, q_2) = p^* \times q_1^* - c_1 q_1^*$$

$$\pi_2(q_1, q_2) = p^* \times q_2^* - c_1 q_2^*$$

Which, after substituting p^*, q_1^* and q_2^*, the profit of firm 1 becomes:

$$\pi_1(q_1, q_2) = \left(\frac{b + c_2 + c_1}{3}\right) \times \left(\frac{b + c_2 - 2c_1}{3a}\right) - c_1 \times \left(\frac{b + c_2 - 2c_1}{3a}\right)$$

And, taking common factor, we get:

$$\pi_1(q_1, q_2) = \left(\frac{b + c_2 - 2c_1}{3a}\right) \times \left(\frac{b + c_2 + c_1}{3} - c_1\right)$$

Multiplying the last term of the second parenthesis by $\frac{3}{3} = 1$, we get:

$$\pi_1(q_1, q_2) = \left(\frac{b + c_2 - 2c_1}{3a}\right) \times \left(\frac{b + c_2 + c_1 - 3c_1}{3}\right)$$

Simplifying, becomes:

$$\pi_1(q_1, q_2) = \left(\frac{b + c_2 - 2c_1}{3a}\right) \times \left(\frac{b + c_2 - 2c_1}{3}\right)$$

Which is equivalent to:

$$\pi_1(q_1, q_2) = \frac{(b + c_2 - 2c_1)^2}{9a}$$

Analogously, after substituting p^*, q_1^* and q_2^*, the profit of firm 2 becomes:

$$\pi_2(q_1, q_2) = \left(\frac{b + c_2 + c_1}{3}\right) \times \left(\frac{b - 2c_2 + c_1}{3a}\right) - c_2 \times \left(\frac{b - 2c_2 + c_1}{3a}\right)$$

And, taking common factor, we get:

$$\pi_2(q_1, q_2) = \left(\frac{b - 2c_2 + c_1}{3a}\right) \times \left(\frac{b + c_2 + c_1}{3} - c_2\right)$$

Multiplying the last term of the second parenthesis by $\frac{3}{3} = 1$, we get:

$$\pi_2(q_1, q_2) = \left(\frac{b - 2c_2 + c_1}{3a}\right) \times \left(\frac{b + c_2 + c_1 - 3c_2}{3}\right)$$

Simplifying, becomes:

$$\pi_2(q_1, q_2) = \left(\frac{b - 2c_2 + c_1}{3a}\right) \times \left(\frac{b + c_1 - 2c_2}{3}\right)$$

Which is equivalent to:

$$\pi_2(q_1, q_2) = \frac{(b + c_1 - 2c_2)^2}{9a}$$

1.3.1. Optimal profit: comparative statics

$$\pi_1(q_1, q_2) = \frac{(b + c_2 - 2c_1)^2}{9a}$$

$$\pi_2(q_1, q_2) = \frac{(b + c_1 - 2c_2)^2}{9a}$$

1.3.1.1. Demand shock:

An increase in demand will inevitably increase the profits of either firm in equilibrium. So, an increase in demand increases quantity, price and a firms' profit. This is intuitive: an increase in demand makes you expect to sell more and at a higher price. Hence, you'd expect to make more profit!

$$\frac{\partial \pi_1(q_1, q_2)}{\partial b} = \frac{2 \times (b + c_2 - 2c_1)}{9a} \geq 0$$
$$\frac{\partial \pi_2(q_1, q_2)}{\partial b} = \frac{2 \times (b + c_1 - 2c_2)}{9a} \geq 0$$

1.3.1.2. Price sensitivity shock:

An increase in consumer's sensitivity to price will reduce the profits of firms in equilibrium. Again, this is intuitive. Recall that an increase in consumer's sensitivity to price reduced the quantity sold in equilibrium and did not change price. So, same price and less quantity means less profit!

$$\frac{\partial \pi_1(q_1, q_2)}{\partial a} = -\frac{(b + c_2 - 2c_1)^2}{9a^2} \leq 0$$
$$\frac{\partial \pi_2(q_1, q_2)}{\partial a} = -\frac{(b + c_1 - 2c_2)^2}{9a^2} \geq 0$$

1.3.1.3. Own cost shock:

An increase in your own unit cost of production may reduce or increase profit, depending on the value of your own cost. This is due to the fact that an increase in own cost reduced the quantity produced but increased price. So, the change in profit will depend on which effect is more important. An increase in own cost will make profits to decrease if the value of your own cost is low enough. As an example, for firm one:

$$\frac{\partial \pi_1(q_1, q_2)}{\partial c_1} = -4 \times \frac{(b + c_2 - 2c_1)}{9a} \leq 0$$

$$-4 \times \frac{(b + c_2)}{9a} + 8\frac{c_1}{9a} \leq 0$$

$$8\frac{c_1}{9a} \leq 4 \times \frac{(b + c_2)}{9a}$$

$$c_1 \leq 4 \times \frac{9a}{8} \times \frac{(b + c_2)}{9a}$$

$$c_1 \leq \frac{4}{8} \times (b + c_2)$$

$$c_1 \leq \frac{(b + c_2)}{2}$$

2. Stackelberg:

2.1. Optimal Quantities:

Firm 2 moves second, and hence we find the same q_2 as before:

$$\frac{b - aq_1 - c_2}{2a} = q_2^*$$

Firm 1 knows firm 2's optimal output, q_2. Hence, he substitutes it into his own profit function:

$$\pi_1(q_1, q_2^*) = bq_1 - aq_1^2 - aq_1 \times \frac{b - aq_1 - c_2}{2a} - c_1q_1$$

Which becomes:

$$\pi_1(q_1, q_2^*) = bq_1 - aq_1^2 - q_1\frac{b - aq_1 - c_2}{2} - c_1q_1$$

Which can be rewritten as:

$$\pi_1(q_1, q_2^*) = bq_1 - aq_1^2 - q_1\frac{b - c_2}{2} + q_1^2\frac{a}{2} - c_1q_1$$

<u>Firm 1 maximises income:</u>

$$\frac{\partial \pi_1(q_1, q_2)}{\partial q_1} = 0 \rightarrow b - 2aq_1 - \frac{b - c_2}{2} + 2q_1\frac{a}{2} - c_1 = 0$$

Simplifying, we get:

$$b - 2aq_1 - \frac{b - c_2}{2} + aq_1 - c_1 = 0$$

Which can be further simplified to:

$$b - aq_1 - \frac{b - c_2}{2} - c_1 = 0$$

Isolating q_1 in the RHS, we get:

$$b - \frac{b - c_2}{2} - c_1 = aq_1$$

Multiplying b and c_1 by $\frac{2}{2}$, we get:

$$\frac{2b - b + c_2 - 2c_1}{2} = aq_1$$

Simplifying, we get:

$$\frac{b + c_2 - 2c_1}{2} = aq_1$$

Dividing both hand sides by a, we get:

$$\frac{b + c_2 - 2c_1}{2a} = q_1^*$$

Substituting q_1^* into q_2, we get:

$$\frac{b - a \times \left(\frac{b + c_2 - 2c_1}{2a}\right) - c_2}{2a} = q_2^*$$

Which can be simplified to:

$$\frac{b-\left(\frac{b+c_2-2c_1}{2}\right)-c_2}{2a}=q_2^*$$

Multiplying the first and last term of the numerator by $\frac{2}{2}=1$, we get:

$$\frac{\frac{2b-b-c_2+2c_1-2c_2}{2}}{2a}=q_2^*$$

Simplifying, we get:

$$\frac{\frac{b-3c_2+2c_1}{2}}{2a}=q_2^*$$

Which can be simplified to:

$$\frac{b-3c_2+2c_1}{4a}=q_2^*$$

2.1.1. Optimal quantities: comparative statics

$$\frac{b + c_2 - 2c_1}{2a} = q_1^*$$

$$\frac{b - 3c_2 + 2c_1}{4a} = q_2^*$$

1.1.1.1. Demand shock:

An increase in b (positive demand shock) will increase the quantities produced of both firms. This is intuitive: the more demand there is the more the firms will produce.

$$\frac{\partial q_1^*}{\partial b} = \frac{1}{2a} \geq 0$$
$$\frac{\partial q_2^*}{\partial b} = \frac{1}{4a} \geq 0$$

However, note that there are two important differences between the Cournot (simultaneous move) vs. the Stackelberg (firm 1 is leader) markets. First, in Cournot the positive demand shock had the same effect in both firms. In Stackelberg, the effect is different in both firms. Before, the effect was $\frac{\partial q_1^*}{\partial b} = \frac{1}{3a}$ for both firms. Now, the effect is bigger $\left(\frac{1}{2a} \geq \frac{1}{3a}\right)$ for the *leader* firm and smaller $\left(\frac{1}{4a} \leq \frac{1}{3a}\right)$ for the *follower* firm.

So, key points to remember are: (a) a positive demand shock influences both firms differently, as opposed to Cournot; (b) a positive demand shock will increase the quantity produced of both firms; (c) a positive demand shock will increase more the quantity produced of the *leader* relative to the *follower* firm; (d) a positive demand shock will increase the quantity of the leader more than it would have under Cournot and will increase the quantity of the follower less than it would have under Cournot.

2.1.1.2. Price sensitivity shock:

An increase in consumers' sensitivity to price will reduce the quantities produced by both firms.

$$\frac{\partial q_1^*}{\partial a} = -\frac{b + c_2 - 2c_1}{2a^2} \leq 0$$
$$\frac{\partial q_2^*}{\partial a} = -\frac{b - 3c_2 + 2c_1}{4a^2} \leq 0$$

However, it will do so differently than under Cournot, which effects are recalled below:

$$\frac{\partial q_1^*}{\partial a} = -\frac{b + c_2 - 2c_1}{3a^2} \leq 0$$

$$\frac{\partial q_2^*}{\partial a} = -\frac{b - 2c_2 + c_1}{3a^2} \leq 0$$

As we can see, for the *leader* the sensitivity towards price as a bigger impact on its quantity produced $\left(-\frac{b+c_2-2c_1}{2a^2} \leq -\frac{b+c_2-2c_1}{3a^2}\right)$, the opposite being true for the *follower* $\left(-\frac{b-3c_2+2c_1}{4a^2} \geq -\frac{b-2c_2+c_1}{3a^2}\right)$. In practice, this means that, for a same increase in the sensitivity to consumer's price, the leader firm will reduce its output more than it does under Stackelberg. The follower firm reduces its output less under Stackelberg than under Cournot.

Together with the previous section (demand shock), we see that there is pros and cons of being a leader. Under a demand shock, and assuming a fixed price, being a leader is more advantageous than Cournot as the increase in quantity is greater. However, being a leader means that, under an increase in consumer's price sensitivity and assuming price remains fixed, being a leader is worse as your output will decrease more under Stackelberg.

2.1.1.3. Competitor's cost shock:

An increase in your competitor's unit cost will increase the quantities produced by both firms, and in the same quantity. Furthermore, it will do so more than under Cournot $\frac{1}{2a} > \frac{1}{3a}$.

$$\frac{\partial q_1^*}{\partial c_2} = \frac{\partial q_2^*}{\partial c_1} = \frac{1}{2a} \geq 0$$

It looks surprising, looking at the asymmetry in Stackelberg between the *leader* and the *follower*, that an increase in the competitor's unit cost will have the same effect on the quantity produced for both the *leader* and the *follower*.

2.1.1.4. Own cost shock:

An increase in your unit cost will decrease the quantities produced by both firms. Furthermore, it will decrease the quantity produced more in the follower $\left(-\frac{1}{a} < -\frac{3}{4a}\right)$.

$$\frac{\partial q_1^*}{\partial c_1} = -\frac{1}{a} \leq 0$$

$$\frac{\partial q_2^*}{\partial c_2} = -\frac{3}{4a} \leq 0$$

Hence, being a leader is worse than being a follower in terms of an increase in one's own cost of production: the reduction in quantities produced is greater and more negative in the leader.

2.2. Optimal Price:

Now, to find the optimal price we substitute q_1^* and q_2^* into $D(Q)$ to get:

$$D(Q) = b - a\left(\frac{b + c_2 - 2c_1}{2a} + \frac{b - 3c_2 + 2c_1}{4a}\right)$$

Multiplying the first element of the parenthesis by $\frac{2}{2} = 1$, we get:

$$D(Q) = b - a\left(\frac{2(b + c_2 - 2c_1) + b - 3c_2 + 2c_1}{4a}\right)$$

Simplifying, we get:

$$D(Q) = b - \frac{2 \times (b + c_2 - 2c_1) + b - 3c_2 + 2c_1}{4}$$

Expanding the parenthesis, we get:

$$D(Q) = b - \frac{2b + 2c_2 - 4c_1 + b - 3c_2 + 2c_1}{4}$$

Simplifying further, we get:

$$D(Q) = b - \frac{3b - c_2 - 2c_1}{4}$$

Multiplying the first element by $\frac{4}{4} = 1$, we get:

$$D(Q) = \frac{4b - 3b + c_2 + 2c_1}{4}$$

Simplifying, we get:

$$D(Q) = \frac{b + c_2 + 2c_1}{4} = p^*$$

2.1.1. Optimal Price: comparative statics

$$D(Q) = \frac{b + c_2 + 2c_1}{4} = p^*$$

2.2.1.1. Positive demand, own cost and competitor's cost shocks:

A positive demand shock and an increase in either your or the other firm's cost will have a positive effect on price. However, and unlike in Cournot, the effect of these three shocks are going to be different:

$$\frac{\partial p^*}{\partial b} = \frac{1}{4} > 0$$

$$\frac{\partial p^*}{\partial c_1} = \frac{1}{2} > 0$$

$$\frac{\partial p^*}{\partial c_2} = \frac{1}{4} > 0$$

We observe that the highest effect is going to come through the leader's unit cost $\left(\frac{1}{2} > \frac{1}{4}\right)$. Also, we observe that the effect of a demand shock and of being a follower are smaller under Stackelberg than under Cournot $\left(\frac{1}{4} < \frac{1}{3}\right)$ and that the effect of an increase in the leader's cost is greater under Stackelberg $\left(\frac{1}{2} > \frac{1}{3}\right)$. Fixing the quantity sold by each firm, all benefit more from a unit cost increase in the leader's good.

2.3. Optimal Profit:

We can write the profit of firms 1 and 2 as:

$$\pi_1(q_1, q_2) = p^* \times q_1^* - c_1 q_1^*$$

$$\pi_2(q_1, q_2) = p^* \times q_2^* - c_1 q_2^*$$

Which, after substituting p^*, q_1^* and q_2^*, the profit of firm 1 becomes:

$$\pi_1(q_1, q_2) = \left(\frac{b + c_2 - 2c_1}{2a}\right) \times \left(\frac{b + c_2 + 2c_1}{4}\right) - \left(\frac{b + c_2 - 2c_1}{2a}\right) \times c_1$$

Taking common factor, we get:

$$\pi_1(q_1, q_2) = \left(\frac{b + c_2 - 2c_1}{2a}\right) \times \left(\frac{b + c_2 + 2c_1}{4} - c_1\right)$$

Multiplying the second element in the second parenthesis by $\frac{4}{4} = 1$, we get:

$$\pi_1(q_1, q_2) = \left(\frac{b + c_2 - 2c_1}{2a}\right) \times \left(\frac{b + c_2 + 2c_1 - 4c_1}{4}\right)$$

Which, after simplifying, becomes:

$$\pi_1(q_1, q_2) = \left(\frac{b + c_2 - 2c_1}{2a}\right) \times \left(\frac{b + c_2 - 2c_1}{4}\right)$$

And, further simplifying, we get:

$$\pi_1(q_1, q_2) = \frac{(b + c_2 - 2c_1)^2}{8a}$$

Similarly for firm 2, after substituting p^*, q_1^* and q_2^*, the profit of firm 2 becomes:

$$\pi_2(q_1, q_2) = \left(\frac{b - 3c_2 + 2c_1}{4a}\right) \times \left(\frac{b + c_2 + 2c_1}{4}\right) - \left(\frac{b - 3c_2 + 2c_1}{4a}\right) \times c_2$$

Taking common factor, we can rewrite the previous expression as:

$$\pi_2(q_1, q_2) = \left(\frac{b - 3c_2 + 2c_1}{4a}\right) \times \left(\frac{b + c_2 + 2c_1}{4} - c_2\right)$$

Multiplying the second element in the second parenthesis by $\frac{4}{4} = 1$, we get:

$$\pi_2(q_1, q_2) = \left(\frac{b - 3c_2 + 2c_1}{4a}\right) \times \left(\frac{b + c_2 + 2c_1 - 4c_2}{4}\right)$$

Which, after simplifying, becomes:

$$\pi_2(q_1, q_2) = \left(\frac{b - 3c_2 + 2c_1}{4a}\right) \times \left(\frac{b - 3c_2 + 2c_1}{4}\right)$$

And, further simplifying, we get:

$$\pi_2(q_1, q_2) = \frac{(b - 3c_2 + 2c_1)^2}{16a}$$

2.3.1. Optimal Profit: comparative statics

$$\pi_1(q_1, q_2) = \frac{(b + c_2 - 2c_1)^2}{8a}$$
$$\pi_2(q_1, q_2) = \frac{(b - 3c_2 + 2c_1)^2}{16a}$$

2.3.1.1. Demand shock:

An increase in demand will have a positive effect on the profits of either firm. Again, this is intuitive as a positive demand shock increases quantity sold and price. The effect is going to be different for each firm:

$$\frac{\partial \pi_1(q_1, q_2)}{\partial b} = \frac{2 \times (b + c_2 - 2c_1)}{4a} \geq 0$$
$$\frac{\partial \pi_2(q_1, q_2)}{\partial b} = \frac{2 \times (b - 3c_2 + 2c_1)}{16a}$$

In the case of the leader, a positive demand shock will have a greater effect in Stackelberg than in Cournot $\left(\frac{2\times(b+c_2-2c_1)}{4a} > \frac{2\times(b+c_2-2c_1)}{9a}\right)$, showing that being a leader is unambiguously more favourable (unfavourable) than Cournot for a positive (negative) demand shock.

2.3.1.2. Price sensitivity shock:

An increase in price sensitivity will have a negative effect on the profit of both firms.

$$\frac{\partial \pi_1(q_1, q_2)}{\partial a} = -\frac{(b + c_2 - 2c_1)^2}{8a^2}$$
$$\frac{\partial \pi_2(q_1, q_2)}{\partial a} = -\frac{(b - 3c_2 + 2c_1)^2}{16a^2}$$

2.3.1.3. Own cost shock:

The effect of an increase in price sensitivity will depend on the value of your own cost. For the leader:

$$\frac{\partial \pi_1(q_1, q_2)}{\partial c_1} = -\frac{(b + c_2 - 2c_1)}{2a} \leq 0$$

$$-\frac{b-c_2}{2a}+\frac{2c_1}{2a}\leq 0$$

$$\frac{2c_1}{2a}\leq\frac{b-c_2}{2a}$$

$$2c_1\leq b-c$$

$$c_1\leq\frac{(b-c_2)}{2}$$

We can see that the cost of the leader needs a lower threshold $\left(\frac{b-c_2}{2}<\frac{b+c_2}{2}\right)$ for the effect of an increase in own cost to yield a decrease in profits.